AU-6234

M.A./M.Sc., (Previous) Examination 2014

Mathematics (First Semester)

TOPOLOGY-I

MODEL ANSWERS

1. a) Definition: Let X be a set and $\mathfrak{J}$ be a collection of sub-sets of X satisfying the following 3 conditions:

T_1: $\phi \in \mathfrak{J}, X \in \mathfrak{J}$; T_2: If $G_1 \in \mathfrak{J}$ and $G_2 \in \mathfrak{J}$, then $G_1 \cap G_2 \in \mathfrak{J}$

T_3: If $G_\lambda \in \mathfrak{J}$ for every $\lambda \in \Lambda$ where Λ is a arbitrary set, then $\cup\{G_\lambda : \lambda \in \Lambda\} \in \mathfrak{J}$.

Example: Any suitable example considered

b) Let $X = \{a, b, c, d, e\}$ and $\mathfrak{J} = \{X, \phi, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}\}$

$A = \{a, c, e\}$

$X \cap A = A$; $\phi \cap A = \phi$; $\{a\} \cap A = \{a\}$; $\{a,b\} \cap A = \{a\}$

$\{a,c,d\} \cap A = \{a,c\}$; $\{a,b,c,d\} \cap A = \{a,c\}$; $\{a,b,e\} \cap A = \{a,e\}$

$\mathfrak{J}_A = \{A, \phi, \{a\}, \{a,c\}, \{a,e\}\}$

c) Let $(X, \mathfrak{J})$ be a topological space and let A be a subset of X. A point $x \in X$ is called a limit point / cluster point of A iff every nbd of x contains a point of A other than x.

d) Let $(\mathbb{R}, U)$ be the usual topological space and let

$F_n = \left[\frac{1}{n}, 1\right]$, $n \in N$ so that F_n is a closed interval of $\mathbb{R}$. F_n is a U-closed set.

Now $\cup\{F_n : n \in N\} = \{1\} \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{4}, 1\right] \cdots = [0, 1]$

is not closed.

(or)

Any suitable example considered.

(e) The exterior points of A is NIL.
(calculation considered)

(f) A fn. $f: X \to Y$ is continuous at $p \in X$ iff, for every open set $H \subset Y$ containing $f(p)$, $f^{-1}(H)$ is a superset of an open set containing p. Hence f is not continuous at $p \in X$ if there exists at least one open set $H \subset Y$ containing $f(p)$ such that $f^{-1}(H)$ doesn't contain an open set containing p.

i.e., $f: X \to Y$ is not continuous at $p \in X$ iff $\exists$ a nbd N of $f(p)$ s.t $f^{-1}(N)$ is not a nbd of p.

g) A subset A of a topological space X is compact if every open cover of A is reducible to a finite cover.
Any suitable related example is considered

h) Let (X, d) be a metric space and $x \in X$.
Consider the collection

$$B(x) = \{ S(x, \tfrac{1}{n}) : n \in N \}$$

If N is any nbd of x, then $\exists\ \varepsilon > 0$ s.t $S(x, \varepsilon) \subset N$
choose n so large that $\frac{1}{n} < \varepsilon$, Then

$$S(x, \tfrac{1}{n}) \subset S(x, \varepsilon) \subset N.$$

Thus, every nbd of x contains a member of $B(x)$ and therefore $B(x)$ is a local base at x; $B(x)$ is countable.
$\therefore X$ is a first countable.

i) A topological space $(X, \mathfrak{J})$ is said to be a T_0-space if and only if given any pair of distinct points x, y of X, there exists a nbd of one of them not containing the other.
Example: Any valid example is considered

(j) Let F_1, F_2 be any pair of disjoint closed sets in a normal space X.
then there exists a continuous map $f: X \to [0,1]$ such that

Q-2 (a) we know that

$$A \subset A \cup B \Rightarrow A' \subset (A \cup B)'$$

$$B \subset A \cup B \Rightarrow B' \subset (A \cup B)'$$

$$\therefore A' \cup B' \subset (A \cup B)' \quad \text{— (1)}$$

Assume $p \notin A' \cup B'$, thus $\exists\, G, H \in \mathcal{J}$ s.t

$p \in G$ and $G \cap A = \{p\}$ and $p \in H$ and $H \cap B \subset \{p\}$

But $G \cap H \in \mathcal{J}$, $p \in G \cap H$ and

$$(G \cap H) \cap (A \cup B) = (G \cap H \cap A) \cup (G \cap H \cap B)' \subset (G \cap A) \cup (H \cap B)$$
$$\subset \{p\} \cup \{p\} = \{p\}$$

Thus $p \notin (A \cup B)'$ and so $(A \cup B)' \subset (A' \cup B')$ — (2)

From (1) and (2) $\quad A' \cup B' = (A \cup B)'$.

(b) suppose A is closed, and let $p \notin A$, i.e. $p \in A^c$. But A^c, the complement of a closed set, is open; hence $p \notin A'$ for A^c is an open set such that

$\cancel{p \notin A}$ $p \in A^c$ and $A^c \cap A = \emptyset$

Thus $A' \subset A$ if A is closed.

Now assume $A' \subset A$; we show that A^c is open. Let $p \in A^c$; then $p \notin A'$, so $\exists$ an open set G such that

$$p \in G \text{ and } (G \setminus \{p\}) \cap A = \emptyset$$

But $p \notin A$; hence $G \cap A = (G \setminus \{p\}) \cap A = \emptyset$

So $G \subset A^c$. Thus p is an interior point of A^c, and so A^c is open.

Q-3 (a) suppose $\mathcal{B}$ is a base for a topology $\mathcal{J}$ on X. Since X is open, X is the union of members of $\mathcal{B}$. Hence X is the union of all the members of $\mathcal{B}$. i.e $X = \cup\{B : B \in \mathcal{B}\}$.

Furthermore, if $B, B^* \in \mathcal{B}$ then, in particular, B and B^* are open. Hence the intersection $B \cap B^*$ is also open and since $\mathcal{B}$ is a base for $\mathcal{J}$, it is the union of members of $\mathcal{B}$. Thus (i) & (ii) of hypothesis are satisfied converse, suppose $\mathcal{B}$ is a class of subsets of X which satisfy i & ii above. Let $\mathcal{T}$ be the class of all subjects of X which are unions of members of $\mathcal{B}$. we claim that $\mathcal{T}$ is a topology on X.

By (i), $X = \cup\{B : B \in \mathcal{B}\}$; so $X \in \mathcal{T}$. Note that ϕ is the union of the empty subclass of $\mathcal{B}$.

i-e $= \cup\{B : B \in \phi \subset \mathcal{B}\}$; hence $\phi \in \mathcal{T}$, and so $\mathcal{T}$ satisfies first condition of topology definition.

Now let $\{G_i\}$ be a class of members of $\mathcal{T}$. By the definition of $\mathcal{T}$, each G_i is the union of members of $\mathcal{B}$; hence the union $\cup_i G_i$ is also the union of members of $\mathcal{B}$ and so belonging to $\mathcal{T}$. Thus $\mathcal{T}$ satisfies second condition of topology.

Lastly, suppose $G, H \in \mathcal{T}$. we need to show that $G \cap H$ also belongs to $\mathcal{T}$. By definition of $\mathcal{T}$, there exists two subclass $\{B_i : i \in I\}$ and $\{B_j : j \in J\}$ of $\mathcal{B}$ such that $G = \cup_i B_i$ and $H = \cup_j B_j$. Than, by the distributive law

$$G \cap H = (\cup_i B_i) \cap (\cup_j B_j) = \cup\{B_i \cap B_j : i \in I, j \in J\}$$

But by ii, $B_i \cap B_j$ is the union of members of $\mathcal{B}$, hence $G \cap H = \cup\{B_i \cap B_j : i \in I, j \in J\}$ is also the union of members of $\mathcal{B}$ and so belongs to $\mathcal{T}$ which therefore satisfies third condition.

(b) Let $\mathcal{B}$ denote the collection of all finite intersections of members of $\mathcal{B}$. Then by actual computation we get

$$\mathcal{B} = \{X, \{a,b\}, \{b,c\}, \{a,d,c\}, \{b\}, \{a\}, \phi\}$$

Note that $X \in \mathcal{B}$ since it is the empty intersection of of members of $\mathcal{B}_*$. Again computing the union of members of $\mathcal{B}$, we get

$$\mathcal{J} = \{X, \{a,b\}, \{b,c\}, \{a,d,e\}, \{b\}, \{a\}, \phi, \{a,b,c\}, \{a,b,d,e\}\}.$$

Q.4 (a) Assume that f is continuous and let F be any closed set in Y. To show that $f^{-1}(F)$ is closed in X. Since f is continuous and $Y-F$ is open in Y, then

$$f^{-1}(Y-F) = f^{-1}(Y) - f^{-1}(F) = X - f^{-1}(F)$$

is open in X. i.e $f^{-1}(F)$ is closed in X.

Converse: Let $f^{-1}(F)$ be closed in X for every closed set F in Y. we want to show that f is a continuous function. Let G be any open set in Y. Then $Y-G$ is closed in Y and so by hypothesis $f^{-1}(Y-G) = X - f^{-1}(G)$ is closed in X, i.e $f^{-1}(G)$ is open in X. Hence f is continuous

(b) The Pasting Lemma: Let $X = A \cup B$, where A and B are closed in X. let $f: A \to Y$ and $g: B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h: X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Proof: Let C be a closed subset of Y. Now

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

Since f is continuous, $f^{-1}(C)$ is closed in A and, therefore closed in X.

Similarly. $g^{-1}(C)$ closed in B and therefore closed in X. Their union $h^{-1}(C)$ is thus closed in X.

Q. 5 Let A be compact relative to X and let $\{V_\lambda : \lambda \in \Lambda\}$ be a collection of sets, open relative to Y, which covers A so that $A \subset \cup\{V_\lambda : \lambda \in \Lambda\}$. Then there exists G_λ, open relative to X, such that $V_\lambda = Y \cap G_\lambda$ for every $\lambda \in \Lambda$. It then follows that

$$A \subset \cup\{G_\lambda : \lambda \in \Lambda\}$$

So that $\{G_\lambda : \lambda \in \Lambda\}$ is an open cover of A relative to X. Since A is compact relative to X, there exists finitely many indices $\lambda_1, \dots, \lambda_n$ such that

$$A \subset G_{\lambda_1} \cup G_{\lambda_2} \cup \dots \cup G_{\lambda_n}.$$

Since $A \subset Y$, we have

$$A \subset Y \cap [G_{\lambda_1} \cup G_{\lambda_2} \cup \dots \cup G_{\lambda_n}]$$
$$= (Y \cap G_{\lambda_1}) \cup \dots \cup (Y \cap G_{\lambda_n})$$

Since $Y \cap G_{\lambda_i} = V_{\lambda_i}$ $(i = 1, 2, \dots, n)$, we get

$$A \subset V_{\lambda_1} \cup \dots \cup V_{\lambda_n}.$$

This shows that A is compact relative to Y.

<u>Converse</u>: Let A be compact relative to Y and let $\{G_\lambda : \lambda \in \Lambda\}$ be a collection of open subsets of X which covers A so that

$$A \subset \cup\{G_\lambda : \lambda \in \Lambda\}$$

Since $A \subset Y$, (1) implies that

$$A \subset Y \cap [\cup\{G_{\lambda_1} : \lambda \in \Lambda\}].$$

Since $Y \cap G_\lambda$ is open relative to Y, the collection

$$\{Y \cap G_\lambda : \lambda \in \Lambda\}$$

is an open cover of A relative to Y, the collection

$$(Y \cap G_\lambda : \lambda \in \Lambda\}$$

is an open cover of A relative to Y. Since A is compact relative to Y, we must have $A \subset (Y \cap G_{\lambda_1}) \cup \dots \cup (Y \cap G_{\lambda_n})$ — (1)

for some choice of finitely many indices, $\lambda_1, \dots, \lambda_n$. But (1)

Q-6(a) Lindelof Theorem:

Every second countable space is Lindelof space

Every closed sub-space of a Lindelof space is Lindelof:

Ans. Let Y be a closed sub-set of a Lindelof space $(X, \mathfrak{J})$. We wish to show that the subspace $(Y, \mathfrak{J}_Y)$ is Lindelof. Let $\{V_\lambda\}$ be any $\mathfrak{J}_Y$-open cover of Y. Then there exists $\mathfrak{J}$-open set G_λ such that $V_\lambda = G_\lambda \cap Y$ so that $\{G_\lambda\}$ is a $\mathfrak{J}$-open cover of Y. The family consisting of all G_λ's and $X-Y$ is then a $\mathfrak{J}$-open cover of X. Since X is Lindelof, this cover has a countable subcover. If we denote this from subcover the set $X-Y$. we obtain countable subcover of Y. Let this subcover be

$$\{G_{\lambda_n} : n \in \Lambda \subset N\}.$$

But then $\{V_{\lambda_n} : n \in \Lambda \subset N\}$ is a countable subcover of the cover $\{V_\lambda\}$ in the subspace $(Y, \mathfrak{J}_Y)$. Hence the sub-space $(Y, \mathfrak{J}_Y)$ is Lindelof.

(b) Let $f : (X, \mathfrak{J}_1) \to (Y, \mathfrak{J}_2)$ be continuous map. Let $A \subset X$ such that A is sequentially compact. To prove that $f(A)$ is sequentially compact. Let ~~Let~~ $\langle y_n \rangle \in f(A)$ be a sequence, $\exists\, x_n \in A$ such that $f(x_n) = y_n$

NOW $\langle x_n \rangle$ is a sequence in A and A is seq. compact. Hence this seq. must have a sub-seq. $\langle x_n \rangle$ such that $x_n \to x \in A$.

By continuity of f, $x_n \to x \in A \Rightarrow f(x_n) \in f(A)$

NOW $\langle f(x_n) \rangle$ is a sub-seq. of $\langle y_n \rangle$ of $f(A)$ converging to $f(x) \in f(A) \Rightarrow f(A)$ is seq. compact.

Q.7 (a) Let $(X, \mathfrak{J})$ be a T_1-space and let f be a one-one mapping to $(X,\mathfrak{J})$ onto another topological-space $(Y, \mathfrak{J}')$. We shall show that $(Y, \mathfrak{J}')$ is also a T_1-space. Let y_1, y_2 be two distinct points of Y. Since f is one-one onto mapping there exists distinct points x_1, x_2 of X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $(X, \mathfrak{J})$ is a T_1-space, there exists $\mathfrak{J}$-open sets G and H such that

$$x_1 \in G \text{ but } x_2 \notin G \text{ and } x_2 \in H \text{ but } x_1 \notin H.$$

Since f is an open mapping, $f(G)$ and $f(H)$ are $\mathfrak{J}'$-open sets such that $y_1 = f(x_1) \in f(G)$ but $y_2 = f(x_2) \notin f(G)$ and $y_2 = f(x_2) \in f(H)$ but $y_1 = f(x_1) \notin f(H)$.

Hence $(Y, \mathfrak{J})$ is also a T_1-space. Since the property of being a T_1-space is preserved under one-one onto open mapping, it is certainly preseved under homemorphism. Hence it is a topological property.

(b) Let A be compact subset of a Hausdorff-space $(X, \mathfrak{J})$. Then to show that $A \subset X$ is closed.

We shall show that A' is open. Let $p \in A'$. Since X is Hausdorff, for every $q \in A$ there exists open nbds of p and q which we denote respectively by $M(q)$ and $N(q)$ such that $M(q) \cap N(q) = \phi$. Now the collection $\{N(q) : q \in A\}$ is an open cover of A. Since A is compact there exists finite number of points, q_i, $i = 1, 2, \dots n$ such that $A \subset \bigcup_{i=1}^{n} N(q_i)$. — ①

Let $M = \cap M(q_i)$, $N = \cup N(q_i)$.

Then M is an open nbd of p. We claim that $M \cap N = \phi$.

We have $x \in N \Rightarrow x \in N(q_i)$ for some i

$\Rightarrow x \notin M(q_i)$

$\Rightarrow x \notin M$

Thus $M \cap N = \phi$ and since $A \subset N$, we have $A \cap M = \phi \Rightarrow M \subset A'$.

Q.8(a) The closed subsets are X, $\{b,c\}$, $\{a\}$, $\emptyset$

Consider the closed set $\{b,c\}$ and the point a not belonging to it. Then $\{b,c\}$ and $\{a\}$ are open sets such that

$$\{b,c\} \subset \{b,c\},\ a \in \{a\} \text{ and } \{b,c\} \cap \{a\} = \emptyset.$$

Similarly consider the closed set $\{a\}$ and the point b not belonging to it. Then $\{a\}$ and $\{b,c\}$ are open sets such that

$$\{a\} \subset \{a\},\ b \in \{b,c\}, \text{ and } \{a\} \cap \{b,c\} = \emptyset.$$

Again for the closed set $\{a\}$ and the point c, there exist open sets $\{a\}$ and $\{b,c\}$ such that $\{a\} \subset \{a\}$, $c \in \{b,c\}$ and $\{a\} \cap \{b,c\} = \emptyset$.

It follows that X is regular space.

Since there does not exist a $\mathfrak{J}$-open set containing the point b and not containing the point c, the space is not T_1-space and consequently it is neither T_2 nor T_3.

(b). Let $(X, \mathfrak{J})$ be a normal space and $(Y, \mathfrak{J}_Y)$ any closed subspace of X. We have to show that $(Y, \mathfrak{J}_Y)$ is also normal. Let L^*, M^* be disjoint $\mathfrak{J}_Y$-closed subsets of Y. Then there exist $\mathfrak{J}$-closed subsets L, M of X such that $L^* = L \cap Y$ and $M^* = M \cap Y$. Since Y is $\mathfrak{J}$-closed it follows that L^*, M^* are disjoint $\mathfrak{J}$-closed subsets of X. Then by normality of X, there exists $\mathfrak{J}$-open subsets G, H of X such that $L^* \subset G$, $M^* \subset H$ and $G \cap H = \emptyset$.

Since $L^* \subset Y$ and $M^* \subset Y$, these relations imply that

$$L^* \subset G \cap Y,\ M^* \subset H \cap Y \text{ and } (G \cap Y) \cap (H \cap Y) = \emptyset$$

Setting $G \cap Y = G^*$ and $H \cap Y = H^*$, we see that G^*, H^* are $\mathfrak{J}_Y$-open subsets of Y such that

$$L^* \subset G^*,\ M^* \subset H^* \text{ and } G^* \cap H^* = \emptyset.$$

Hence $(Y, \mathfrak{J}_Y)$ is a normal space.